Pergamon

0021-8928(95)00021-6

J. Appl. Maths Mechs, Vol. 59, No. 2, pp. 175–185, 1995

0021-8928/95 $24.00+0.00

THE DYNAMICS OF THE PRECESSIONAL MOTIONS OF A SYSTEM OF TWO RIGID BODIES IN A GRAVITATIONAL FIELD†

I. Ye. BIRMAN and G. V. GORR

Donetsk

(*Received 2 March* 1994)

The conditions for the existence of precessions in a system consisting of two symmetric rigid bodies in a gravitational field are derived, on the assumption that one of the bodies is rotating uniformly about the vertical and the other is precessing about the vertical. Regular precession of asymmetric bodies linked by a spherical hinge is considered.

A great many classes of precessional motions, in various fields of force, have been determined in the dynamics of a single rigid body.‡ In the classical problem, the characteristic properties that mark a motion as precessional are conditions imposed on the mass distribution, such as the Lagrange, Hess and Grioli conditions, and the requirement that the barycentric axis of the body should make a fixed angle with the axis of precession. In the case of more general fields of force, these properties may fail to hold.‡ In the problem of the motion of a system of coupled rigid bodies in a gravitational field, conditions have been derived for the existence of regular precessions of Lagrange gyroscopes [1] and semi-regular precessions of Hess gyroscopes [2], and certain properties of precessional motions of systems of two coupled rigid bodies in a gravitational field have been established in the case when one of the bodies is a Lagrange or Hess gyroscope and the other is a Grioli gyroscope [3].

All the studies cited are based on conditions of a specific mass distribution in the system.

The present paper will discuss precessions of a system of two rigid bodies in a gravitational field without any prior assumptions as to the mass distribution of the bodies. Special attention will be devoted to cases in which one of the bodies is rotating uniformly about the vertical. It is obvious that if a body S_2 is suspended in a body S_1 at its centre of mass, it will rotate as a free rigid body. In that case the dynamics of the precessions of S_1 is the same as in the classical case. This special case will not be considered here.

The studies presented here of the dynamics of precessions in a system of two heavy non-symmetric rigid bodies complement results previously obtained for precession with prescribed mass distributions [1–3] and show that some of the properties established are analogous to those known to exist in the classical problem. However, along with facts typical for the classical case, we also obtain new facts; this is particularly true in regard to uniform motions in the system of two heavy bodies considered here.

1. FORMULATION OF THE PROBLEM

Consider the motion of a system of two rigid bodies S_1 and S_2 with the following characteristics: O_1 is a fixed point of S_1; the letter O will denote a general point of either S_1 or S_2, at which there is an ideal spherical hinge; C_1 and C_2 denote the mass centres of S_1 and S_2, and m_1 and m_2 denote their masses. Then the equation of motion of the system consisting of S_1 and S_2 in a gravitational field, assuming that there is no friction at O_1 and O_2, are as follows [1]:

$$A_1\dot{\boldsymbol{\omega}}_1+\boldsymbol{\omega}_1\times A_1\boldsymbol{\omega}_1+m_2\mathbf{s}\times\left[\frac{d\mathbf{v}_0}{dt}+\frac{d}{dt}(\boldsymbol{\omega}_2\times\mathbf{e}_2)-g\boldsymbol{\nu}\right]-m_1g\mathbf{e}_1\times\boldsymbol{\nu}=0 \tag{1.1}$$

$$A_2\overset{*}{\boldsymbol{\omega}}_2+\boldsymbol{\omega}_2\times A_2\boldsymbol{\omega}_2+m_2\mathbf{e}_2\times(d\mathbf{v}_0/dt-g\boldsymbol{\nu})=0 \tag{1.2}$$

$$\dot{\boldsymbol{\nu}}=\boldsymbol{\nu}\times\boldsymbol{\omega}_1,\quad \overset{*}{\boldsymbol{\nu}}=\boldsymbol{\nu}\times\boldsymbol{\omega}_2 \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 59, No. 2, pp. 188–198, 1995.

‡GORR G. V., Precessional motions in rigid body dynamics and the dynamics of systems of coupled rigid bodies. Preprint No. 89.03, Donetsk, Inst. Prikl. Mat. Mekh. Akad. Nauk UkrSSR, 1989.

For these formulae we have introduced the following notation: $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are the absolute angular velocity vectors of S_1 and S_2, $\boldsymbol{\nu}$ is a unit vector indicating the direction of the force of gravity, $\mathbf{v}_0$ is the velocity of the point O, $\mathbf{e}_1 = \mathbf{O_1C_1}$, $\mathbf{e}_2 = \mathbf{OC_2}$, $\mathbf{s} = \mathbf{O_1O}$, A_1 and A_2 are the inertia tensors of S_1 and S_2, which are constant at the points O_1 and O, g is the acceleration due to gravity, and a dot and an asterisk over a variable denotes relative derivatives in bases $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ and $\mathbf{э}_1$, $\mathbf{э}_2$, $\mathbf{э}_3$ rigidly attached to the bodies S_1 and S_2. The following relations are obvious

$$\mathbf{v}_0 = \frac{d\mathbf{s}}{dt} = \boldsymbol{\omega}_1 \times \mathbf{s}, \quad \frac{d\mathbf{v}_0}{dt} = \dot{\boldsymbol{\omega}}_1 \times \mathbf{s} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{s})$$
$$\frac{d\mathbf{e}_1}{dt} = \boldsymbol{\omega}_1 \times \mathbf{e}_1, \quad \frac{d\mathbf{e}_2}{dt} = \boldsymbol{\omega}_2 \times \mathbf{e}_2 \tag{1.4}$$

Equations (1.1)–(1.3) admit of first integrals

$$\boldsymbol{\nu} \cdot [A_1\boldsymbol{\omega}_1 + A_2\boldsymbol{\omega}_2 + m_2(\mathbf{e}_2 + \mathbf{s}) \times \mathbf{v}_0 + m_2\mathbf{s} \times (\boldsymbol{\omega}_2 \times \mathbf{e}_2)] = k$$
$$(A_1\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1) + (A_2\boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_2) + m_2 v_0^2 + 2m_2\mathbf{v}_0 \cdot (\boldsymbol{\omega}_2 \times \mathbf{e}_2) -$$
$$-2m_1 g(\mathbf{e}_1 \cdot \boldsymbol{\nu}) - 2m_2 g(\mathbf{s} + \mathbf{e}_2) \cdot \boldsymbol{\nu} = 2E, \quad \boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1 \tag{1.5}$$

Let

$$\boldsymbol{\omega}_1 = \sum_{j=1}^{3} p_j^{(1)} \mathbf{i}_j, \quad \boldsymbol{\omega}_2 = \sum_{k=1}^{3} p_k^{(2)} \mathbf{э}_k, \quad \boldsymbol{\nu} = \sum_{n=1}^{3} \nu_n^{(1)} \mathbf{i}_n \tag{1.6}$$

$$\mathbf{э}_i = \sum_{j=1}^{3} \alpha_{ij} \mathbf{i}_j, \quad \mathbf{i}_l = \sum_{n=1}^{3} \beta_{ln} \mathbf{э}_n, \quad (i = 1,2,3; \; l = 1,2,3) \tag{1.7}$$

where the product of the matrices (α_{ij}), (β_{ln}) is obviously equal to the identity matrix. Let $O_1\xi\eta\zeta$ denote a fixed frame of reference whose unit vectors are $\mathbf{i}, \mathbf{j}, \mathbf{k} = \boldsymbol{\nu}$. The positions of the bases of S_1 and S_2 relative to $O_1\xi\eta\zeta$ are determined by the Euler angles θ_1, φ_1, ψ_1 and θ_2, φ_2, ψ_2 and the matrices $(\gamma_{ij}^{(1)})$, $(\gamma_{ij}^{(2)})$, respectively, where $\theta_1 = \angle(\boldsymbol{\nu}, \mathbf{i}_3)$, $\theta_2 = \angle(\boldsymbol{\nu}, \mathbf{э}_3)$; the position of the basis of S_2 relative to S_1 is determined by the angles θ, φ, ψ, where $\theta = \angle(\mathbf{i}_3\, \mathbf{э}_3)$. Then

$$\gamma_{11}^{(i)} = \cos\varphi_i \cos\psi_i - \cos\theta_i \sin\varphi_i \sin\psi_i$$
$$\gamma_{12}^{(i)} = \cos\varphi_i \sin\psi_i + \cos\theta_i \sin\varphi_i \cos\psi_i, \quad \gamma_{13}^{(i)} = \sin\varphi_i \sin\theta_i$$
$$\gamma_{21}^{(i)} = -\sin\varphi_i \cos\psi_i - \cos\theta_i \sin\psi_i \cos\varphi_i \tag{1.8}$$
$$\gamma_{22}^{(i)} = -\sin\varphi_i \sin\psi_i + \cos\theta_i \cos\varphi_i \cos\psi_i, \quad \gamma_{23}^{(i)} = \cos\varphi_i \sin\theta_i$$
$$\gamma_{31}^{(i)} = \sin\theta_i \sin\psi_i, \quad \gamma_{32}^{(i)} = -\sin\theta_i \cos\psi_i, \quad \gamma_{33}^{(i)} = \cos\theta_i, \quad i = 1,2$$

where

$$(\gamma_{ij}^{(2)}) = (\gamma_{il}^{(1)})(\alpha_{lj})$$
$$\alpha_{11} = \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi, \quad \alpha_{12} = \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi$$
$$\alpha_{13} = \sin\varphi\sin\theta, \quad \alpha_{21} = -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi \tag{1.9}$$
$$\alpha_{22} = -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi, \quad \alpha_{23} = \cos\varphi\sin\theta$$
$$\alpha_{31} = \sin\theta\sin\psi, \quad \alpha_{32} = -\sin\theta\cos\psi, \quad \alpha_{33} = \cos\theta$$

For the absolute angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ we have

$$p_1^{(i)} = \dot{\psi}_i \sin\theta_i \sin\varphi_i + \dot{\theta}_i \cos\varphi_i, \quad p_2^{(i)} = \dot{\psi}_i \sin\theta_i \cos\varphi_i - \dot{\theta}_i \sin\varphi_i$$
$$p_3^{(i)} = \dot{\psi}_i \cos\theta_i + \dot{\varphi}_i, \quad (i = 1,2) \tag{1.10}$$

Let $\boldsymbol{\omega}_*$ be the angular velocity of S_2 relative to S_1. Then

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_* \tag{1.11}$$

where

$$\boldsymbol{\omega}_* = (\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)\boldsymbol{\ni}_1 + (\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi)\boldsymbol{\ni}_2 + (\dot{\psi}\cos\theta + \dot{\varphi})\boldsymbol{\ni}_3 \tag{1.12}$$

Using (1.6)–(1.12), we can write system (1.1) and (1.2) in terms of the variables θ_1, φ_1, ψ_1 and θ, φ, ψ. However, this yields a rather cumbersome system of differential equations, not very suitable for investigating the special class of precessions. In this paper, therefore, we shall adopt another approach, based in the first instance on a basic property of precessional motions. After deriving the conditions for such motions to exist under our assumptions, we shall then indicate the positions of the bases of S_1, S_2 relative to $O_1\xi\eta\zeta$ and relative to one another, relying on the aforementioned kinematic characteristics.

2. PRECESSIONAL MOTIONS

Suppose that each of S_1 and S_2 is precessing. This means that one can find in S_1 and S_2 fixed unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, and in the space $O_1\xi\eta\zeta$ unit vectors $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, such that the angles between $\mathbf{a}_1$ and $\boldsymbol{\gamma}_1$ and $\mathbf{a}_2$ and $\boldsymbol{\gamma}_2$, respectively, remain constant throughout the time of motion. This property may be written as follows:

$$\mathbf{a}_1 \cdot \boldsymbol{\gamma}_1 = a_0^{(1)}, \quad \mathbf{a}_2 \cdot \boldsymbol{\gamma}_2 = a_0^{(2)}, \quad \dot{\boldsymbol{\gamma}}_1 = \boldsymbol{\gamma}_1 \times \boldsymbol{\omega}_1, \quad \overset{*}{\boldsymbol{\gamma}}_2 = \boldsymbol{\gamma}_2 \times \boldsymbol{\omega}_2 \tag{2.1}$$

We then obtain from (2.1)

$$\boldsymbol{\omega}_1 = u_1(t)\mathbf{a}_1 + \upsilon_1(t)\boldsymbol{\gamma}_1, \quad \boldsymbol{\omega}_2 = u_2(t)\mathbf{a}_2 + \upsilon_2(t)\boldsymbol{\gamma}_2 \tag{2.2}$$

If $u_1(t)$ and $\upsilon_1(t)$ do not depend on time, the precession of S_1 is said to be regular. When one of these functions is constant, the precession is said to be semi-regular. If neither $u_1(t)$ nor $\upsilon_1(t)$ is a constant, we have the most general kind of precession. The definitions of precession for S_2 are analogous. In addition, if $\boldsymbol{\gamma}_1 = \boldsymbol{\nu}$, we shall call the motion a precession of S_1 about the vertical; otherwise it will be a precession of S_1 about an inclined axis. A detailed survey of the results obtained for precession in the dynamics of a single rigid body may be found in the preprint cited above. We will merely note that in the classical case—the motion of a body in a field of gravity—the possible cases are regular precession of a Lagrange gyroscope, semi-regular precession of a Hess gyroscope about the vertical, and regular precession of a Grioli gyroscope about an inclined axis. One of the characteristic properties is that the vector $\mathbf{a}_i$ points along the barycentric axis. In systems dynamics this property is also typical of many precessions; we shall therefore confine our attention to precessions of this type only.

Suppose that S_1 is precessing and let κ_1 be the angle between the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_1$. We shall assume that $\mathbf{i}_3 = \mathbf{a}_1$ and that θ_1 is the angle between the vectors $\boldsymbol{\gamma}_1$ and $\mathbf{i}_3$. Then it is obvious that $a_0^{(1)} = \cos\theta_1$ and $\theta_1 = \theta_0^{(1)}$, where $\theta_0^{(1)}$ is a constant. Substituting $\boldsymbol{\omega}_1 = u_1(t)\mathbf{i}_3 + \upsilon_1(t)\boldsymbol{\gamma}_1$ into the equation for the derivatives of $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_1$ from (1.3) and (2.1), we obtain

$$\dot{\boldsymbol{\gamma}}_1 = u_1(t)(\boldsymbol{\gamma}_1 \times \mathbf{i}_3), \quad \dot{\boldsymbol{\nu}} = u_1(t)(\boldsymbol{\nu} \times \mathbf{i}_3) + \upsilon_1(t)(\boldsymbol{\nu} \times \boldsymbol{\gamma}_1) \tag{2.3}$$

Noting that $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_1$ are unit vectors and stipulating that ψ_1 is the angle of precession with the axis of precession along the vector $\boldsymbol{\gamma}_1$, we obtain the following representations for the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}_1$ from (2.3)

$$\boldsymbol{\gamma}_1 = a_{01}^{(1)}\sin\varphi_1\mathbf{i}_1 + a_{01}^{(1)}\cos\varphi_1\mathbf{i}_2 + a_0^{(1)}\mathbf{i}_3 \tag{2.4}$$

$$\boldsymbol{\nu} = (c_0 + b_0' a_0^{(1)}\sin\psi_1)\boldsymbol{\gamma}_1 - b_0'\sin\psi_1\mathbf{i}_3 - b_0'(\boldsymbol{\gamma}_1 \times \mathbf{i}_3)\cos\psi_1 \tag{2.5}$$

where φ_1 is the angle of rotation of the body about an axis along the vector $\mathbf{i}_3$, $1a_{01}^{(1)} = \sin\theta_0^{(1)}$, $c_0 = \cos\kappa_1$, $b_0' = \sin\kappa_1/\sin\theta_0^{(1)}$. In this situation $u_1(t) = \dot{\varphi}_1$, $\upsilon_1(t) = \dot{\psi}_1$, i.e. it follows from (2.2) that

$$\boldsymbol{\omega}_1 = \dot{\varphi}_1 \mathbf{a}_1 + \dot{\psi}_1 \boldsymbol{\gamma}_1 \tag{2.6}$$

This approach enables us to express the vectors $\boldsymbol{\gamma}_1$, $\boldsymbol{\nu}$, $\boldsymbol{\omega}_1$ explicitly in terms of the corresponding variables and to consider only the dynamic equations of motion of S_1. The same holds for the dynamics of precessions of S_2. Note that the variable ψ_1 introduced here differs from the corresponding variable of Section 2 only in the case of precessions about an inclined axis. We have therefore used the same notation ψ_1 in order to avoid introducing new symbols.

In this paper we are concerned with precessions of a system of two coupled bodies S_1 and S_2 on the assumption that both S_1 and S_2 precess about the vertical, i.e. $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \boldsymbol{\nu}$. In addition, as already remarked, we shall assume that the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ point along the barycentric axes of S_1 and S_2 and the mass centre of S_1 is situated on the line-segment $[O_1O]$. Let $\mathbf{s} = s\mathbf{i}_3$, $\mathbf{e}_1 = e_1\mathbf{i}_3$, $\mathbf{e}_2 = e_2\mathbf{э}_3$ in Eqs (1.1) and (1.2). Then, by (1.4), (2.2) and the assumptions just made, it follows from (1.1) and (1.2) that

$$\begin{aligned} &A_1\dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times A_1\boldsymbol{\omega}_1 + \mathbf{i}_3 \times [P_2(\dot{\boldsymbol{\omega}}_1 \times \mathbf{i}_3 + \boldsymbol{\omega}_1(\boldsymbol{\omega}_1 \cdot \mathbf{i}_3) - \mathbf{i}_3\omega_1^2) + \\ &+P_1(\overset{*}{\boldsymbol{\omega}}_2 \times \mathbf{э}_3 + \boldsymbol{\omega}_2(\boldsymbol{\omega}_2 \cdot \mathbf{э}_3) - \omega_2^2\mathbf{э}_3) - \Gamma_3\boldsymbol{\nu}] - \Gamma_1(\mathbf{i}_3 \times \boldsymbol{\nu}) = 0 \end{aligned} \tag{2.7}$$

$$A_2\overset{*}{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times A_2\boldsymbol{\omega}_2 + \mathbf{э}_3 \times [P_1(\dot{\boldsymbol{\omega}}_1 \times \mathbf{i}_3 + \boldsymbol{\omega}_1(\boldsymbol{\omega}_1 \cdot \mathbf{i}_3) - \omega_1^2\mathbf{i}_3) - \Gamma_2\boldsymbol{\nu}] = 0 \tag{2.8}$$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\nu} \times \boldsymbol{\omega}_1, \quad \overset{*}{\boldsymbol{\nu}} = \boldsymbol{\nu} \times \boldsymbol{\omega}_2 \tag{2.9}$$

The integrals (1.5) are

$$\begin{aligned} &\boldsymbol{\nu} \cdot [A_1\boldsymbol{\omega}_1 + A_2\boldsymbol{\omega}_2 + (P_1\mathbf{э}_3 + P_2\mathbf{i}_3) \times (\boldsymbol{\omega}_1 \times \mathbf{i}_3) + P_1\mathbf{i}_3 \times (\boldsymbol{\omega}_2 \times \mathbf{э}_3)] = k, \quad \boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1 \\ &(A_1\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1) + (A_2\boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_2) + P_2(\boldsymbol{\omega}_1 \times \mathbf{i}_3)^2 \times 2P_1(\boldsymbol{\omega}_1 \times \mathbf{i}_3) \cdot (\boldsymbol{\omega}_2 \times \mathbf{э}_3) - \\ &-2\Gamma_1(\mathbf{i}_3 \cdot \boldsymbol{\nu}) - 2(\Gamma_3\mathbf{i}_3 + \Gamma_2\mathbf{э}_3) \cdot \boldsymbol{\nu} = 2E \end{aligned} \tag{2.10}$$

In all these formulae we have used the notation $P_1 = m_2e_2s$, $P_2 = m_2s^2$, $\Gamma_1 = m_1e_1g$, $\Gamma_2 = m_2e_2g$, $\Gamma_3 = m_2sg$.

Let us assume that S_1 and S_2 are precessing about the vertical, under the above restrictions. We then derive from (2.1) and (2.2)

$$\begin{aligned} &\mathbf{i}_3 \cdot \boldsymbol{\nu} = a_0^{(1)}, \quad \mathbf{э}_3 \cdot \boldsymbol{\nu} = a_0^{(2)} \\ &\boldsymbol{\omega}_1 = u_1(t)\mathbf{i}_3 + \upsilon_1(t)\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = u_2(t)\mathbf{э}_3 + \upsilon_2(t)\boldsymbol{\nu} \end{aligned} \tag{2.11}$$

Inserting $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ into (2.7) and (2.8), we obtain

$$\begin{aligned} &\dot{u}_1(t)A_1\mathbf{i}_3 + \dot{\upsilon}_1(t)A_1\boldsymbol{\nu} + u_1(t)\upsilon_1(t)[\mathrm{tr}(A_1)(\boldsymbol{\nu} \times \mathbf{i}_3) - 2(A_1\boldsymbol{\nu} \times \mathbf{i}_3)] + \\ &+\dot{u}_1^2(t)(\mathbf{i}_3 \times A_1\mathbf{i}_3) + \dot{\upsilon}_1^2(t)(\boldsymbol{\nu} \times A_1\boldsymbol{\nu}) + \mathbf{i}_3 \times [P_2\dot{\upsilon}_1(t)(\boldsymbol{\nu} \times \mathbf{i}_3) + P_1\dot{\upsilon}_2(t)(\boldsymbol{\nu} \times \mathbf{э}_3) + \\ &+(P_2\upsilon_1^2(t)a_0^{(1)} + P_1\upsilon_2^2(t)a_0^{(2)} - \Gamma_3)\boldsymbol{\nu} - P_1\upsilon_2^2(t)\mathbf{э}_3] - \Gamma_1(\mathbf{i}_3 \times \boldsymbol{\nu}) = 0 \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\dot{u}_2(t)A_2\mathbf{э}_3 + \dot{\upsilon}_2(t)A_2\boldsymbol{\nu} + u_2(t)\upsilon_2(t)[\mathrm{tr}(A_2)(\boldsymbol{\nu} \times \mathbf{э}_3) - 2(A_2\boldsymbol{\nu} \times \mathbf{э}_3)] + \\ &+u_2^2(t)(\mathbf{э}_3 \times A_2\mathbf{э}_3) + \upsilon_2^2(t)(\boldsymbol{\nu} \times A_2\boldsymbol{\nu}) + \mathbf{э}_3 \times [P_1\dot{\upsilon}_1(t)(\boldsymbol{\nu} \times \mathbf{i}_3) + \\ &+P_1\upsilon_1^2(t)a_0^{(1)} - \Gamma_2)\boldsymbol{\nu} - P_1\upsilon_1^2(t)\mathbf{i}_3] = 0 \end{aligned} \tag{2.13}$$

$$\dot{\boldsymbol{\nu}} = u_1(t)(\boldsymbol{\nu} \times \mathbf{i}_3), \quad \overset{*}{\boldsymbol{\nu}} = u_2(t)(\boldsymbol{\nu} \times \mathbf{э}_3) \tag{2.14}$$

were tr (A_i) are the traces of the matrices A_1 and A_2: $A_1 = (A_{ij}), A_2 = (B_{ij})$.

For the absolute derivatives of the vectors $\mathbf{i}_3$ and $\mathbf{э}_3$ we have, as in the case of (1.4)

$$\frac{d\mathbf{i}_3}{dt} = \upsilon_1(t)(\boldsymbol{\nu} \times \mathbf{i}_3), \quad \frac{d\mathbf{э}_3}{dt} = \upsilon_2(t)(\boldsymbol{\nu} \times \mathbf{э}_3) \tag{2.15}$$

To investigate the conditions for precessions of a system of two coupled bodies S_1 and S_2 to exist, one has to consider special cases in which one of the bodies is spinning uniformly [3]. These cases are also of interest in themselves.

3. THE FIRST TYPE OF MOTION OF THE SYSTEM

Suppose that S_1 is precessing regularly and that S_2 is spinning uniformly about the barycentric axis directed along the vertical. Then by (2.11)

$$\mathbf{i}_3 \cdot \boldsymbol{\nu} = a_0^{(1)}, \quad a_0^{(2)} = 1, \quad \boldsymbol{\omega}_1 = n_0 \mathbf{i}_3 + m_0 \boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = m_0' \boldsymbol{\vartheta}_3 \tag{3.1}$$

that is, $u_1(t) = n_0$, $\upsilon_1(t) = m_0$, $u_2(t) = m_0'$, $\upsilon_2(t) = 0$. Substituting these values into (2.12), (2.13) and the first equation of (2.14), we get

$$n_0 m_0 [\operatorname{tr}(A_1)(\boldsymbol{\nu} \times \mathbf{i}_3) - 2(A_1 \boldsymbol{\nu} \times \mathbf{i}_3)] + n_0^2 (\mathbf{i}_3 \times A_1 \mathbf{i}_3) + m_0^2 (\boldsymbol{\nu} \times A_1 \boldsymbol{\nu}) +$$
$$+ (P_2 m_0^2 a_0^{(1)} - \Gamma_3 - \Gamma_1)(\mathbf{i}_3 \times \boldsymbol{\nu}) = 0 \tag{3.2}$$

$$m_0'^2 (\boldsymbol{\vartheta}_3 \times A_2 \boldsymbol{\vartheta}_3) - P_1 m_0^2 (\boldsymbol{\vartheta}_3 \times \mathbf{i}_3) = 0, \tag{3.3}$$

$$\dot{\boldsymbol{\nu}} = n_0 (\boldsymbol{\nu} \times \mathbf{i}_3) \tag{3.4}$$

Equation (3.4) yields a representation for $\boldsymbol{\nu}$

$$\boldsymbol{\nu} = a_{01}^{(1)} \sin \varphi_1 \mathbf{i}_1 + a_{01}^{(1)} \cos \varphi_1 \mathbf{i}_2 + a_0^{(1)} \mathbf{i}_3 \tag{3.5}$$

where $a_{01}^{(1)} = \sqrt{(1 - a_0^{(1)2})}$, $\varphi_1 = n_0 t + \varphi_1^{(0)}$. Let $\mathbf{i}_3 = \beta_{31}\boldsymbol{\vartheta}_1 + \beta_{32}\boldsymbol{\vartheta}_2 + \beta_{33}\boldsymbol{\vartheta}_3$ (see (1.7)). Then, taking into account that $A_2 = (B_{ij})$, we deduce from Eq. (3.3) that

$$P_1 m_0^2 \beta_{32} - m_0'^2 B_{23} = 0, \quad P_1 m_0^2 \beta_{31} - m_0'^2 B_{13} = 0 \tag{3.6}$$

Hence it follows that β_{31} and β_{32} are constants, and since $\beta_{31}^2 + \beta_{32}^2 + \beta_{33}^2 = 1$, β_{33} is also a constant. Hence the vector $\mathbf{i}_3$ is fixed in the basis of S_2. Using the first equation of (2.15) and the fact that β_{31}, β_{32}, β_{33} are constants, it can be shown that $m_0' = m_0$, and it then follows from (3.6) that

$$\beta_{31} = B_{13} / P_1, \quad \beta_{32} = B_{23} / P_1, \quad \beta_{33} = \cos \theta_1^{(0)} = (1 - (B_{13}^2 + B_{23}^2) / P_1^2]^{1/2}$$

Substituting (3.5) into Eq. (3.2) and noting that the resulting equation must be an identity in t, we obtain the conditions

$$\begin{gathered} A_{12} = A_{13} = A_{23} = 0, \quad A_{11} = A_{22} \\ m_0 n_0 A_{33} + m_0^2 a_0^{(1)} (A_{33} - A_{11}) - P_2 m_0^2 a_0^{(1)} + \Gamma_3 + \Gamma_1 = 0 \end{gathered} \tag{3.7}$$

These formulae show that S_1 is a Lagrange gyroscope. Let

$$\beta_{31} = a_{01}^{(1)} \cos \alpha_0, \quad \beta_{32} = a_{01}^{(1)} \sin \alpha_0, \quad \beta_{33} = a_0^{(1)}, \quad \sin \alpha_0 = \frac{B_{23}}{P_1 a_{01}^{(1)}}, \quad \cos \alpha_0 = \frac{B_{13}}{P_1 a_{01}^{(1)}}$$

Then the conditions on the parameters of the body may be written as

$$B_{13}^2 + B_{23}^2 = (a_{01}^{(1)} P_1)^2 \tag{3.8}$$

The position of S_2 relative to S_1 is given by the first formula of (1.7), with

$$\alpha_{11} = -a_0^{(1)} \cos \alpha_0 \sin \varphi_1 - \sin \alpha_0 \cos \varphi_1$$
$$\alpha_{12} = -a_0^{(1)} \cos \alpha_0 \cos \varphi_1 + \sin \alpha_0 \sin \varphi_1$$
$$\alpha_{13} = a_0^{(1)} \cos \alpha_0, \quad \alpha_{21} = -a_0^{(1)} \sin \alpha_0 \sin \varphi_1 + \cos \alpha_0 \cos \varphi_1$$
$$\alpha_{22} = -a_0^{(1)} \sin \alpha_0 \cos \varphi_1 - \cos \alpha_0 \sin \varphi_1, \quad \alpha_{23} = a_0^{(1)} \sin \alpha_0$$
$$\alpha_{31} = a_{01}^{(1)} \sin \varphi_1, \quad \alpha_{32} = a_{01}^{(1)} \cos \varphi_1, \quad \alpha_{33} = a_0^{(1)}$$

For the first type of motion the solution of Eqs (1.1)–(1.3) is

$$\boldsymbol{\omega}_1 = n_0\mathbf{i}_3 + m_0\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = m_0\text{э}_3$$
$$\boldsymbol{\nu} = a_{01}^{(1)} \sin\varphi_1\mathbf{i}_1 + a_{01}^{(1)} \cos\varphi_1\mathbf{i}_2 + a_0^{(1)}\mathbf{i}_3, \quad \varphi_1 = n_0 t + \varphi_1^{(0)}$$

The position of S_1 relative to the fixed system of coordinates $O_1\xi\eta\zeta$ is defined by the matrix $(\gamma_{ij}^{(1)})$, whose components are given by formulae (1.8) with $i = 1$ and $\theta_1 = \theta_1^{(0)}$, $\varphi_1 = n_0 t + \varphi_1^{(0)}$, $\psi_1 = m_0 t + \psi_1^{(0)}$.

If one assumes that S_2 is also a Lagrange gyroscope, it follows from (3.8) that it must be suspended from S_1 at its centre of mass. Consequently, in the general case S_2 cannot be symmetric about its axis of spin. Thus, it is characteristic for the first type of motion that S_1 is a Lagrange gyroscope but S_2 is not, and its axis of spin cannot be a principal axis of S_2.

4. THE SECOND TYPE OF MOTION OF THE SYSTEM

Suppose that S_1 is precessing regularly and that S_2 is spinning uniformly about a non-barycentric axis directed along the vertical

$$\mathbf{i}_3 \cdot \boldsymbol{\nu} = a_0^{(1)}, \quad a_0^{(2)} = 1, \quad \boldsymbol{\omega}_1 = n_0\mathbf{i}_3 + m_0\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = m_0'\mathbf{b} \tag{4.1}$$

where $\mathbf{b} \neq \text{э}_3$, $\mathbf{b} = \boldsymbol{\nu}$. Then the equation of motion of S_2, by (2.13), is

$$m_0'^2(\mathbf{b} \times A_2\mathbf{b}) + \text{э}_3 \times [(P_1 m_0^2 a_0^{(1)} - \Gamma_2)\boldsymbol{\nu} - P_1 m_0^2 \mathbf{i}_3] = 0 \tag{4.2}$$

Let us provide S_2 with an intermediate basis э_1^*, э_2^*, $\text{э}_3^* = \mathbf{b}$ such that $\text{э}_3 = \gamma_1\text{э}_1^* + \gamma_3\text{э}_3^*$. Define

$$\mathbf{i}_3 = \beta_{31}^*\text{э}_1^* + \beta_{32}^*\text{э}_2^* + \beta_{33}^*\text{э}_3^* \tag{4.3}$$

By (4.3), it follows from Eq. (4.2) that

$$\begin{gathered} P_1\gamma_3 m_0^2\beta_{32}^* - m_0'^2 B_{23}^* = 0, \quad \gamma_1\beta_{32}^* = 0 \\ m_0'^2 B_{13}^* - \gamma_1(P_1 m_0^2 a_0^{(1)} - \Gamma_2 - P_1 m_0^2\beta_{33}^*) - P_1\gamma_3 m_0^2\beta_{31}^* = 0 \end{gathered} \tag{4.4}$$

Since $\gamma_1 \neq 0$, it follows that $\beta_{32}^* = 0$, $B_{23}^* = 0$. In addition, it follows from (2.15), when conditions (4.1)–(4.4) are satisfied, that $m_0' = m_0$, $\beta_{33}^* = a_0^{(1)}$, $\beta_{31}^* = a_0^{(1)}$. Let $\gamma_1 = \cos\alpha_0^*$, $\gamma_3 = \sin\alpha_0^*$. Then by (4.4)

$$m_0^2 B_{13}^* + \Gamma_2\cos\alpha_0^* - P_1 m_0^2 a_0^{(1)}\sin\alpha_0^* = 0 \tag{4.5}$$

If the axis of spin is a principal axis of S_2, it follows from (4.5) that $\mathrm{tg}\,\alpha_0^* = \Gamma_2/P_1 m_0^2 a_0^{(1)}$. When conditions (3.5), (4.1)–(4.4) are satisfied, Eq. (2.12) yields the first two conditions of (3.7).

Thus, in the second type of motion S_1 is again a Lagrange gyroscope. The difference between this and the first type is that S_2 may spin uniformly about a principal axis in S_2 at O, which, however, cannot be a barycentric axis. The position of the bases of S_1 and S_2 is easily determined from the relationships indicated in Sections 3 and 4.

5. THE THIRD TYPE OF MOTION OF THE SYSTEM

Suppose that S_1 is spinning uniformly about the vertical and that S_2 is precessing regularly

$$\boldsymbol{\omega}_1 = m_0\boldsymbol{\nu}, \quad \mathbf{i}_3 \cdot \boldsymbol{\nu} = a_0^{(1)}, \quad \text{э}_3 \cdot \boldsymbol{\nu} = a_0^{(2)}, \quad \boldsymbol{\omega}_2 = n_0'\text{э}_3 + m_0'\boldsymbol{\nu} \tag{5.1}$$

The vector $\boldsymbol{\nu}$ is fixed in S_1. Denote it by $\mathbf{a}_1$. We may assume that $\mathbf{a}_1$ is not the vector $\mathbf{i}_3$. We provide S_1 with a basis $\mathbf{i}_1^*$, $\mathbf{i}_2^*$, $\mathbf{i}_3^* = \mathbf{a}_1$, setting

$$\mathbf{i}_3 = \delta_{31}\mathbf{i}_1^* + \delta_{33}\mathbf{i}_3^*, \quad \text{э}_3 = \sigma_{31}\mathbf{i}_1^* + \sigma_{32}\mathbf{i}_2^* + \sigma_{33}\mathbf{i}_3^* \tag{5.2}$$

Analysis of Eq. (2.12) under assumptions (5.1) and (5.2) yields the conditions

$$\delta_{31}\sigma_{32} = 0, \quad P_1\sigma_{32}\delta_{33} - A_{23}^* = 0$$

$$A_{13}^{*}m_0^2 - \delta_{31}(P_2 m_0^2\delta_{33} + P_1 m_0'^2 a_0^{(2)} - \Gamma_3 - \Gamma_1) + P_1 m_0'^2(\delta_{31}\sigma_{33} - \delta_{33}\sigma_{31}) = 0 \tag{5.3}$$

If $\delta_{31} = 0$, i.e. S_1 is spinning uniformly about a barycentric axis, then $\delta_{33} = 1$ and by (5.3)

$$P_1\sigma_{32} = A_{13}, \quad P_1\sigma_{31} = A_{23} \tag{5.4}$$

It has been taken into account here that $m_0' = m_0$. This condition follows from the equation

$$d\mathbf{э}_3 / dt = m_0'(\boldsymbol{\nu} \times \mathbf{э}_3)$$

taking into account the fact that the σ_{ij} in (5.4) are constants. It follows from (5.4) that if the barycentric axis of S_1 is a principal axis, then S_2 is also spinning uniformly. This means that a necessary condition for the existence of a precession of S_2 is that the barycentric axis should not be a principal axis of S_1.

If $\delta_{31} \neq 0$ in (5.3), then $\sigma_{32} = 0, A_{23}^{*} = 0$. Then one can set $\sigma_{33} = a_0^{(2)}$, $\sigma_{31} = a_{01}^{(2)}$, and we obtain from (5.3)

$$A_{13}^{*}m_0^2 - m_0^2 P_2\delta_{31}\delta_{33} + \Gamma_1\delta_{31} - P_1 m_0^2\delta_{33}a_0^{(2)} = 0 \tag{5.5}$$

Examination of Eq. (2.13) yields the conditions

$$B_{12} = B_{13} = B_{23} = 0, \quad B_{11} = B_{22} \tag{5.6}$$

$$B_{33}a_{01}^{(2)}n_0'm_0 - (B_{11} - B_{33})a_0^{(2)}a_{01}^{(2)}m_0^2 - a_{01}^{(2)^2}(m_0^2 P_1\delta_{33} - \Gamma_2) +$$
$$+ P_1 m_0^2[\sigma_{33} - a_0^{(2)}(\delta_{31}\sigma_{31} + \delta_{33}\sigma_{33})] = 0 \tag{5.7}$$

Thus, it follows from (5.6) that S_2 is a Lagrange gyroscope. The conditions for this type of motion to exist have the form (5.5) and (5.7). The solution for the third type is

$$\boldsymbol{\omega}_1 = m_0\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = n_0'\mathbf{э}_3 + m_0\boldsymbol{\nu}$$
$$\boldsymbol{\nu} = a_{01}^{(2)}\sin\varphi_2\mathbf{э}_1 + a_{01}^{(2)}\cos\varphi_2\mathbf{э}_2 + a_0^{(2)}\mathbf{э}_3, \quad \varphi_2 = n_0't + \varphi_2^{(0)}$$

6. THE FOURTH TYPE OF MOTION OF THE SYSTEM

Suppose that S_1 is performing semi-regular precession of the first type about the vertical, while S_2 is spinning uniformly about $\mathbf{b} = \boldsymbol{\nu}$. We must set

$$\boldsymbol{\omega}_1 = u_1(t)\mathbf{i}_3 + m_0\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = m_0'\boldsymbol{\nu} \tag{6.1}$$

in (2.11). When this is done, Eq. (2.13) yields (4.2) and the analysis follows the same lines as in Section 4. Noting that, as is obvious, $m_0' = m_0$ in (6.1), we deduce from Eq. (2.12) that

$$\dot{u}_1(t)A_1\mathbf{i}_3 + u_1(t)m_0[\mathrm{tr}(A_1)(\boldsymbol{\nu} \times \mathbf{i}_3) - 2(A_1\boldsymbol{\nu} \times \mathbf{i}_3)] + u_1^2(t)(\mathbf{i}_3 \times A_1\mathbf{i}_3) +$$
$$+ m_0^2(\boldsymbol{\nu} \times A_1\boldsymbol{\nu}) + (\mathbf{i}_3 \times \boldsymbol{\nu})(P_2 m_0^2 a_0^{(1)} - \Gamma_3 - \Gamma_1) + P_1 m_0^2 a_0^{(2)}(\mathbf{i}_3 \times \boldsymbol{\nu}) - P_1 m_0^2(\mathbf{i}_3 \times \mathbf{э}_3) = 0 \tag{6.2}$$

We obtain an expansion (3.5) for the vector $\boldsymbol{\nu}$, where $u_1 = \dot{\varphi}_1$. It follows from the arguments of Section 4 that in this case $\boldsymbol{\nu} \cdot (\mathbf{э}_3 \times \mathbf{i}_3) = 0$.

Let us calculate the scalar product of the left-hand side of (6.2) and the vectors $\mathbf{i}_3$ and $\boldsymbol{\nu}$, respectively

$$\ddot{\varphi}_1(A_1\mathbf{i}_3 \cdot \mathbf{i}_3) + m_0^2\mathbf{i}_3 \cdot (\boldsymbol{\nu} \times A_1\boldsymbol{\nu}) = 0$$
$$\ddot{\varphi}_1(A_1\mathbf{i}_3 \cdot \boldsymbol{\nu}) - 2\dot{\varphi}_1 m_0\boldsymbol{\nu} \cdot (A_1\boldsymbol{\nu} \times \mathbf{i}_3) + \dot{\varphi}_1^2\boldsymbol{\nu} \cdot (\mathbf{i}_3 \times A_1\mathbf{i}_3) = 0 \tag{6.3}$$

The first equation of (6.3) in scalar form yields

$$\dot{\varphi}_1^2(t) = \frac{a_{01}^{(1)}m_0^2}{2A_{33}}[a_{01}^{(1)}(A_{22} - A_{11})\cos 2\varphi_1 + 2a_{01}^{(1)}A_{12}\sin 2\varphi_1 +$$
$$+ 4A_{23}a_0^{(1)}\cos\varphi_1 + 4A_{13}a_0^{(1)}\sin\varphi_1 + c_*] \tag{6.4}$$

where c_* is an arbitrary constant.

Substituting (6.4) into the second equation of (6.3) we obtain the following conditions on the parameters

$$A_{12} = A_{23} = 0, \quad A_{13}^2 = A_{33}(A_{11} - A_{22}) \tag{6.5}$$

and the relation

$$\dot{\varphi}_1 = -\frac{m_0}{A_{33}}(A_{13}a_{01}^{(1)} \sin\varphi_1 + A_{33}a_0^{(1)}) \tag{6.6}$$

Examining the projection of the left-hand side of Eq. (6.2) onto the vector $\mathbf{i}_3 \times \boldsymbol{\nu}$, we obtain the condition

$$\begin{aligned} &A_{22}a_{01}^{(1)}a_0^{(1)^2} m_0^2 + a_{01}^{(1)^2}(P_2 m_0^2 a_0^{(1)} - \Gamma_3 - \Gamma_1) + P_1 m_0^2 a_{01}^{(1)^2} a_0^{(2)} - \\ &- P_1 m_0^2 [a_0^{(2)} - a_0^{(1)}(\mathbf{i}_3 \cdot \mathbf{э}_3)] = 0 \end{aligned} \tag{6.7}$$

Formulae (6.5)–(6.7) imply that this motion of a coupled system of two bodies S_1 and S_2 is possible only if S_1 if a Hess gyroscope, $\dot{\varphi}_1$ is determined by (6.6), the properties of the uniform motion of S_2 are analogous to those given in Section 6, and the parameters of S_1 and S_2 satisfy condition (6.7). This result for S_1 is similar to the result for the classical problem of a single rigid body with a fixed point.

7. THE DYNAMIC IMPOSSIBILITY OF SEMI-REGULAR PRECESSION OF THE SECOND TYPE

By virtue of the analogy established in Section 6 with the classical problem of a single rigid body, it is interesting to investigate semi-regular motions of the second type. Thus, let S_1 perform a semi-regular precession of the second type and S_2 spin uniformly about the vertical

$$\boldsymbol{\omega}_1 = n_0\mathbf{i}_3 + \upsilon(t)\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = m_0'\mathbf{b}, \quad \mathbf{i}_3 \cdot \boldsymbol{\nu} = a_0^{(1)}, \quad \mathbf{b} = \boldsymbol{\nu} \tag{7.1}$$

Equation (2.13) becomes

$$m_0'^2(\mathbf{b} \times A_2\mathbf{b}) + \mathbf{э}_3 \times [P_1\dot{\upsilon}_1(t)(\mathbf{b} \times \mathbf{i}_3) + P_1\upsilon_1^2(t)a_0^{(1)} - \Gamma_2)\mathbf{b} - P_1\upsilon_1^2(t)\mathbf{i}_3 = 0 \tag{7.2}$$

We introduce a basis $\mathbf{э}_1^*$, $\mathbf{э}_2^*$, $\mathbf{э}_3^* = \mathbf{b}$ in S_2 so that

$$\mathbf{э}_3 = \sin\alpha_0\mathbf{э}_1^* + \cos\alpha_0\mathbf{э}_3^*, \quad \mathbf{i}_3 = a_{01}^{(1)}\sin u\mathbf{э}_1^* + a_{01}^{(1)}\cos u\mathbf{э}_2^* + a_0^{(1)}\mathbf{э}_3 \tag{7.3}$$

where u is an auxiliary variable. Let us consider the kinematics of this motion. Obviously

$$\frac{d\mathbf{i}_3}{dt} = \upsilon_1(t)(\boldsymbol{\nu} \times \mathbf{i}_3), \quad \frac{d\mathbf{э}_1^*}{dt} = m_0'\mathbf{э}_2^*, \quad \frac{d\mathbf{э}_2^*}{dt} = -m_0'\mathbf{э}_1^* \tag{7.4}$$

Substituting (7.3) into the first equation of (7.4), we have

$$\upsilon_1(t) = m_0' - \dot{u} \tag{7.5}$$

Equation (7.2) in scalar notation gives the equalities

$$\begin{aligned} &\sin\alpha_0(\dot{\upsilon}_1(t)\sin u - \upsilon_1^2\cos u) = 0 \\ &m_0'^2 B_{23}^* + P_1 a_0^{(1)}\cos\alpha_0(\dot{\upsilon}_1(t)\sin u - \upsilon_1^2(t)\cos u) = 0 \\ &m_0'^2 B_{13}^* - P_1 a_0^{(1)}\cos\alpha_0(\dot{\upsilon}_1\cos u - \upsilon_1^2\sin u) + \Gamma_2\sin\alpha_0 = 0 \end{aligned} \tag{7.6}$$

If $\sin\alpha_0 = 0$ in (7.6), it follows from these relations that

$$m_0'^2(B_{13}^*\sin u + B_{23}^*\cos u) - P_1 a_{01}^{(1)}\dot{\upsilon}_1(t) = 0$$
$$m_0''^2(B_{23}^*\sin u - B_{13}^*\cos u) + P_1 a_{01}^{(1)}\dot{\upsilon}_1(t) = 0 \qquad (7.7)$$

When u = const, it follows from (7.3) that $\upsilon_2(t) = m_0'$, i.e. the precession of S_1 cannot be semi-regular. Differentiating the first relationship of (7.7) along the trajectories of the second, we obtain

$$(B_{13}^*\cos u + B_{23}^*\sin u)(\dot{u} - 2\upsilon_1(t)) = 0$$

The case $B_{13}^* = B_{23}^* = 0$ is impossible because of (7.7). Therefore $u = 2\upsilon_1(t)$ and it follows from (7.5) that only regular precession of S_1 is possible. Thus, in system (7.6) $\sin\alpha_0 \neq 0$, $B_{23}^* = 0$ and

$$\dot{\upsilon}_1(t)\sin u - \upsilon_1^2(t)\cos u = 0, \quad \upsilon_1^2(t) = \kappa_0 \sin u \qquad (7.8)$$

where

$$\kappa_0 = (m_0'^2 B_{13}^* + \Gamma_2 \sin\alpha_0)/P_1 a_{01}^{(1)}\cos\alpha_0 \qquad (7.9)$$

Let us assume that $\cos\alpha \neq 0$. In that case it follows from the two equations of (7.8) (with the second written as $2\upsilon_1(t)\dot{\upsilon}_1(t) = \kappa_0\dot{u}\cos u$) and from Eq. (7.5) that $\dot{u} = 2/3m_0'$, i.e. u is a constant. Thus, we must assume in (7.6) that $\cos\alpha_0 = 0$, and so system (7.6) yields equalities

$$B_{23}^* = 0, \quad B_{13}^* = -\frac{\Gamma_2}{m_0'}, \quad \dot{\upsilon}_1(t)\sin u - \upsilon_1^2(t)\cos u = 0 \qquad (7.10)$$

Under these conditions, because of (3.5), the integrals (2.10) become

$$\upsilon_1^2(t)(a_2\sin 2\varphi_1 + a_2'\cos 2\varphi_1 + 2a_1\sin\varphi_1 + 2a_1'\cos\varphi_1 + a_0) +$$
$$+2\upsilon_1(t)[P_1 m_0' a_{01}^{(1)}\sin u + n_0(b_1\sin\varphi_1 + b_1'\cos\varphi_1 + b_0)] = 2E^*$$
$$\upsilon_1(t)[(a_2\sin 2\varphi_1 + a_2'\cos 2\varphi_1 + a_1\sin\varphi_1 + a_1'\cos\varphi_1 + a_0^*) +$$
$$+P_1 a_{01}^{(1)}\sin u] + P_1 m_0' a_{01}^{(1)}\sin u + n_0(b_1\sin\varphi_1 + b_1'\cos\varphi_1 + b_0) = k^* \qquad (7.11)$$

where E^*, k^* are new constants and

$$a_2 = A_{12}a_{01}^{(1)^2}, \quad a_2' = \tfrac{1}{2}(A_{22} - A_{11})a_{01}^{(1)^2}, \quad a_1 = A_{13}a_0^{(1)}a_{01}^{(1)}$$
$$a_1' = A_{23}a_0^{(1)}a_{01}^{(1)}, \quad a_0^* = A_{33}a_0^{(1)^2} + \tfrac{1}{2}(A_{11} + A_{22})a_{01}^{(1)^2}$$
$$a_0 = a_0^* + m_2 s^2 a_{01}^{(1)^2}, \quad b_1 = A_{13}a_{01}^{(1)}, \quad b_1' = A_{23}a_{01}^{(1)}, \quad b_0 = A_{33}a_0^{(1)}$$

Under conditions (3.5) and (7.1), projection of the left-hand side of Eq. (2.12) onto the vector $\mathbf{i}_3$ gives

$$\dot{\upsilon}_1(t)(b_1\sin\varphi_1 + b_1'\cos\varphi_1 + b_0) = \upsilon_1^2(t)(a_2\cos 2\varphi_1 - a_2'\sin 2\varphi_1 + a_1\cos\varphi_1 - a_1'\sin\varphi_1) \qquad (7.12)$$

where $\varphi_1 = n_0 t + \varphi_1^{(0)}$.

We substitute (7.5) into the last equation of (7.10) and make the change $z = \operatorname{ctg} u$ in the resulting equation. Then

$$\ddot{z}(1+z^2) - \dot{z}^2 z + 2m_0' z\dot{z}(1+z^2) + m_0'^2(1+z^2)^2 = 0 \qquad (7.13)$$

Comparing (7.10) and (7.12), we have

$$z(t) = P_2(\varphi_1)/P_1(\varphi_1) \qquad (7.14)$$
$$P_2(\varphi_1) = a_2\cos 2\varphi_1 - a_2'\sin 2\varphi_1 + a_1\cos\varphi_1 - a_1'\sin\varphi_1$$
$$P_1(\varphi_1) = b_1\sin\varphi_1 + b_1'\cos\varphi_1 + b_0$$

We substitute (7.14) into Eq. (7.13) and require that the resulting equation be an identity with respect to φ_1. The special case thus obtained is characterized by the conditions $A_{12} = A_{13} = A_{23} = 0, A_{11} = A_{22}, a_0^{(1)} = 0$ which, on the basis of the integrals (7.11), yield a regular precession. Excluding this case, we see that the substitution gives $a_2 = 0, a_2' = 0, b_1 = 0, b_1' = 0, a_1 = 0, a_1' = 0$. But this means that S_1 is precessing regularly. We have thus proved that type of motion 5, for which the first body is performing a semi-regular precession of the second type and the second, S_2 is spinning uniformly about the vertical, is impossible.

Note that a motion in which S_1 is spinning uniformly about the vertical, while S_2 is performing semi-regular precession, is also dynamically impossible.

8. THE CASE OF REGULAR PRECESSION

It has been shown [1] that Lagrange gyroscopes in a system of n coupled bodies admit of regular precession. However, the research in question assumed a priori that all the bodies in the system are Lagrange gyroscopes. We shall not make this assumption here, and we shall consider the regular precession of two bodies S_1 and S_2 in the general case.

Let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be a basis in S_1 and $\mathbf{э}_1, \mathbf{э}_2, \mathbf{э}_3$ a basis in S_2. In addition, set

$$\mathbf{i}_3 = \beta_{31}\mathbf{э}_1 + \beta_{32}\mathbf{э}_2 + \beta_{33}\mathbf{э}_3 \tag{8.1}$$

$$\boldsymbol{\omega}_1 = n_0\mathbf{i}_3 + m_0\boldsymbol{\nu}, \quad \boldsymbol{\omega}_2 = n_0'\mathbf{э}_3 + m_0'\boldsymbol{\nu}, \quad \mathbf{i}_3\cdot\boldsymbol{\nu} = a_0^{(1)}, \quad \mathbf{э}_3\cdot\boldsymbol{\nu} = a_0^{(2)} \tag{8.2}$$

It follows from system (2.12), (2.13), on the basis of (8.1) and (8.2) and provided that

$$\boldsymbol{\nu} = a_{01}^{(1)}\sin\varphi_1\mathbf{i}_1 + a_{01}^{(1)}\cos\varphi_1\mathbf{i}_2 + a_0^{(1)}\mathbf{i}_3$$
$$\boldsymbol{\nu} = a_{01}^{(2)}\sin\varphi_2\mathbf{э}_1 + a_{01}^{(2)}\cos\varphi_2\mathbf{э}_2 + a_0^{(2)}\mathbf{э}_3$$

for the body S_1

$$\begin{aligned}
&A_{12} = 0, \quad A_{11} = A_{22}, \quad A_{13}a_0^{(1)} = 0, \quad A_{23}a_0^{(1)} = 0\\
&n_0^2(c_1\cos\varphi_1 - c_1'\sin\varphi_1) + P_1a_{01}^{(2)}m_0^2\sin u\sin(\varphi_2 - \upsilon) = 0\\
&m_0n_0a_0^* + n_0^2(c_1\sin\varphi_1 + c_1'\cos\varphi_1) + m_0^2(g_1\sin\varphi_1 + g_1'\cos\varphi_1 + g_0) + P_2a_0^{(1)}a_{01}^{(1)}m_0^2 -\\
&-P_1a_0^{(2)}a_{01}^{(1)^2}m_0'^2 - \Gamma_1a_{02}^{(1)^2} + P_1a_0^{(1)}m_0'^2\beta_{33} = 0
\end{aligned} \tag{8.3}$$

and for the body S_2

$$\begin{aligned}
&B_{12} = 0, \quad B_{11} = B_{22}, \quad B_{13}a_0^{(2)} = 0, \quad B_{23}a_0^{(2)} = 0\\
&n_0'^2(d_1\cos\varphi_2 - d_1'\sin\varphi_2) - P_1m_0^2a_{01}^{(2)}\sin u\sin(\varphi_2 - \upsilon) = 0\\
&m_0'^2a_{01}^{(1)^2}[a_0^{(1)}(B_{11} - B_{33}) - B_{13}a_{01}^{(2)}\sin\varphi_2 - B_{23}a_{01}^{(2)}\cos\varphi_2] +\\
&+m_0'n_0'b_0^* + n_0'^2(d_1\sin\varphi_2 + d_1'\cos\varphi_2) + a_{01}^{(1)^2}(P_2a_0^{(1)}m_0 - \Gamma_3) - P_1m_0^2(a_0^{(1)} - \beta_{33}a_0^{(2)}) = 0
\end{aligned} \tag{8.4}$$

where

$$g_0 = \tfrac{1}{2}a_0^{(1)}a_{01}^{(1)^2}(A_{11} + A_{22} - 2A_{33}), \quad b_0^* = B_{33}a_0^{(1)^2} + \tfrac{1}{2}(B_{11} + B_{22})a_{01}^{(1)^2},$$
$$c_1 = A_{13}a_{01}^{(1)}, \quad c_1' = A_{23}a_{01}^{(1)}, \quad g_1 = -a_{01}^{(1)^3}A_{13}, \quad g_1' = -a_{01}^{(1)^3}A_{23}, \quad d_1 = B_{13}a_{01}^{(2)},$$
$$d_1' = B_{23}a_{01}^{(1)}, \quad \beta_{31} = \sin u\sin\upsilon, \quad \beta_{32} = \sin u\cos\upsilon, \quad \beta_{33} = \cos u$$

(u and υ are new variables). If we assume that $a_0^{(1)} = a_0^{(2)} = 0$, then it is obvious that $\sin u = 0$, and therefore $A_{13} = A_{23} = 0$ and $B_{13} = B_{23} = 0$. When $a_0^{(1)} = a_0^{(2)} = 0$, it follows from (8.3), (8.4) that $B_{13} = B_{23} = 0$, i.e. $\sin u \sin(\varphi_2 - \upsilon) = 0$. But this means that also $A_{13} = A_{23} = 0$. A similar conclusion is reached in the case when $a_0^{(1)} = \neq 0, a_0^{(2)} \neq 0$. Thus, in any case, both bodies are Lagrange gyroscopes. In that case

u reduces to a constant, and it is easy to derive the final conditions for regular precessions to exist from (8.3) and (8.4). This result complements that obtained by Kharlamov [1].

REFERENCES

1. KHARLAMOV P. V., The equations of motion of a system of rigid bodies. In *Mechanics of Solids*, No. 4, pp. 52–73, Naukova Dumka, Kiev, 1972.
2. GORR G. V. and RUBANOVSKII V. N., A class of motions of a system of heavy rigid bodies coupled by hinges. *Prikl. Mat. Mekh.* **52**, 5, 707–712, 1988.
3. GORR G. V. and BIRMAN I. Ye., The precessional motions of a coupled system of two rigid bodies in a field of gravity. In *Mechanics of Solids*, No. 24, pp. 56–61, Naukova Dumka, Kiev, 1992.

Translated by D.L.